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NON-INDUCED ISOMORPHISMS OF MATRIX RINGS

BY

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ABSTRACT

We give examples of distinct integers i, j and rings T for which the matrix rings $M_i(T)$ and $M_j(T)$ are isomorphic as rings, but for which the free modules ${}_TT^{(i)}$ and ${}_TT^{(j)}$ are non-isomorphic as T-modules.

Throughout this note $M_i(R)$ denotes the ring of $i \times i$ matrices over the ring R, while $R^{(i)}$ denotes the direct sum of i copies of the left regular module ${}_RR$. All rings are assumed to have a unit element, all subrings are assumed to have the same unit element as the including ring, and all ring homomorphisms are understood to preserve the unit element.

There are many known examples of rings R for which there is a ring isomorphism $M_i(R) \cong M_j(R)$ with $i \neq j$. For instance, if $R = RFM(S)$ (the ring of countably infinite row-finite matrices over the ring S), then $M_i(R) \cong M_j(R)$ for all integers i and j. All such isomorphisms between $M_i(R)$ and $M_j(R)$ for $i \neq j$ which appear in the literature are induced by starting with a ring R for which $R^{(i)} \cong R^{(j)}$ as left R-modules, and then taking endomorphism rings: $M_i(R) \cong \mathrm{End}_R(R^{(i)}) \cong \mathrm{End}_R(R^{(j)}) \cong M_j(R)$. We answer in the affirmative the following question: is it possible to find a ring T and two distinct integers i, j with the property that $M_i(T) \cong M_j(T)$ but the left T-modules $T^{(i)}$ and $T^{(j)}$ are NOT isomorphic? This of course implies that the matrix ring isomorphism is not induced by an isomorphism of free modules as described above.

We present below, in great detail, a procedure which produces a ring T for which $T = M_1(T) \cong M_2(T)$, but for which $T^{(1)}$ is not isomorphic to $T^{(2)}$. A

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somewhat less detailed generalization of this procedure will then be described so as to yield the following

PROPOSITION: *For any pair of integers $n > 1$ and $m \geq 1$ there exists a ring T for which $M_m(T) \cong M_{mn}(T)$ as rings, but $T^{(m)}$ is not isomorphic to $T^{(mn)}$ as T-modules.*

Let R be any ring for which there is an isomorphism ϕ: $M_4(R) \to R = M_1(R)$. (There are many such rings; for instance, any ring for which $R^{(4)}$ is isomorphic to $R^{(1)}$ as left R-modules. Such rings include any ring of the form $R = RFM(S)$. But later we will need additional information about R which will preclude us from using rings of this particular form.) Let ι: $M_2(R) \to M_4(R)$ denote the usual embedding as scalars of a ring of 2 by 2 matrices into a ring of 4 by 4 matrices. Let ψ denote the composition $\phi \circ \iota$; so ψ: $M_2(R) \to R$. But since ϕ is an isomorphism and ι is not surjective we have that the image of ψ is not all of R. We denote $\mathrm{Im}(\psi)$ by S_1, and we denote by

$$\psi_1\colon M_2(R) \to S_1$$

the ring isomorphism gotten by viewing the codomain of ψ as S_1.

But S_1 is a proper subring of R, so $M_2(S_1)$ is a proper subring of $M_2(R)$. Let S_2 denote the image of $M_2(S_1)$ under ψ_1. So we have an isomorphism

$$\psi_2\colon M_2(S_1) \to S_2.$$

As above we have that S_2 is a proper subring of S_1, so that $M_2(S_2)$ is a proper subring of $M_2(S_1)$. Upon repeating the process we produce the following sequence of inclusions, where all the vertical maps are isomorphisms:

$$\begin{array}{ccccc} S_1 & \supset & S_2 & \supset & S_3\cdots \\ \uparrow{\scriptstyle \psi_1} & & \uparrow{\scriptstyle \psi_2} & & \uparrow{\scriptstyle \psi_3} \\ M_2(R) & \supset & M_2(S_1) & \supset & M_2(S_2)\cdots \end{array}$$

Now let T denote $\bigcap_{i \in \mathbf{N}} S_i$ (the intersection of the rings in the top row), and let U denote $\bigcap_{i \in \mathbf{N}} M_2(S_i)$ (the intersection of the rings in the bottom row). It is trivial to show that $U = M_2(T)$. We let $\widehat{\psi}$: $U \to T$ be the function defined by restriction of the ψ_i; $\widehat{\psi}$ is well-defined (since each ψ_{i+1} is defined as the restriction of ψ_i for each i), and $\widehat{\psi}$ is an isomorphism. So we have

$$M_1(T) = T \cong U = M_2(T).$$

Thus T is a 'candidate' for a ring of the desired type.

We now show that we can pick the original ring R in such a way that the left T-modules $T = T^{(1)}$ and $T^{(2)}$ are not isomorphic. To do so, we use some ideas of Leavitt [2]. The **module type** (or simply **type**) of a ring R is defined as follows. If R has invariant basis number (i.e., for all integers i and j, $R^{(i)} \cong R^{(j)}$ if and only if $i = j$), then the **type** of R is defined to be the letter d (for **dimensional**). Otherwise, there exists a smallest integer n with the property that $R^{(n)}$ is isomorphic to some $R^{(v)}$ with $v > n$. Pick v minimal with this property and let k denote $v - n$. In this case we say that R has **type** (n_R, k_R), or simply **type** (n, k). By [2, Theorem 1], if R has type (n, k), then for any two integers y and z each greater than or equal to n we have $R^{(y)} \cong R^{(z)}$ if and only if $y \equiv z (\operatorname{mod} k)$. In particular, $R^{(1)} \cong R^{(2)}$ if and only if R has type $(1, 1)$. Furthermore, it is shown in [2] that: (1) For an arbitrary pair $(n, k) \in \mathbf{N} \times \mathbf{N}$ there exists a ring R whose type is (n, k), (2) If A, A' are rings with types (n, k) and (n', k') respectively, and if there exists a ring homomorphism from A to A' (e.g. if A is a subring of A'), then $n' \leq n$ and $k' \mid k$, and (3) If R has type (n, k) then for any integer m the ring $M_m(R)$ has type

$$\left(\frac{n+r}{m}, \frac{k}{\gcd(k,m)}\right)$$

where r is the smallest nonnegative integer such that $m \mid n + r$.

Now let R be a ring of type (1,3); such exists by statement (1). Then $R^{(4)} \cong R^{(1)}$, since $4 \equiv 1 \pmod 3$. We use this module isomorphism to induce a ring isomorphism ϕ: $M_4(R) \to R$, and in turn produce the intersection ring T, as described above. But $T \subseteq S_1 \subseteq R$, so by statement (2) we have that the type of T is (n_T, k_T) where $3 \mid k_T$. In particular, the type of T is not (1,1), so that

$$T^{(1)} \text{ is not isomorphic to } T^{(2)}.$$

This completes the detailed description of the particular example.

To prove the Proposition, we generalize the above procedure as follows. Let $n > 1$ and $m \geq 1$ be arbitrary integers, and let $l = m^2 n$. Let R be a ring of type $(1, mn + 1)$; so notationally we have $k_R = mn + 1$. Then $R^{(nlm)} \cong R^{(m)}$, since $nlm = m^3 n^2 = m(mn-1)(mn+1)+m \equiv m \pmod{k_R}$. This module isomorphism induces an isomorphism of matrix rings ϕ: $M_{nlm}(R) \to M_m(R)$, which we view as ϕ: $M_{nl}(M_m(R)) \to M_m(R)$. We define ι: $M_n(M_m(R)) \to M_{nl}(M_m(R))$ to be the usual embedding as scalars, define $\psi = \phi \circ \iota \colon M_n(M_m(R)) \to M_m(R)$, and let S_1 denote $\mathrm{Im}(\psi)$. We now repeat the procedure described above (with $M_m(R)$ in the role of R) to produce an intersection ring T with the property that $T \cong M_n(T)$. This isomorphism yields an isomorphism

$$M_m(T) \cong M_m(M_n(T)) \cong M_{mn}(T).$$

Since $\gcd(m, k_R) = 1$ we have by statement (3) that $k_R = k_{M_m(R)}$, which in turn yields (as $T \subseteq S_1 \subseteq M_m(R)$) that $k_R \mid k_T$. In particular this gives $k_T \geq k_R = mn + 1 > mn - m$, so that $mn \not\equiv m \pmod{k_T}$, from which we conclude that

$$T^{(m)} \text{ is not isomorphic to } T^{(mn)}.$$

With this, T has been shown to be a ring having the properties set forth in the Proposition.

We conclude with four remarks. First, an intersection procedure similar to the one employed here has also been used in the context of infinite dimensional matrix rings; see [1]. Second, we do not know whether the statement $k_R \mid k_T$ of the previous paragraph can be strengthened to the statement $k_R = k_T$. Third, we do not know whether for arbitrary positive integers $i \neq j$ there exists a ring T for which $M_i(T) \cong M_j(T)$, but for which $T^{(i)}$ is not isomorphic to $T^{(j)}$.

Finally, suppose R is a ring for which $R^{(z)} \cong R^{(nz)}$; then we get an isomorphism of matrix rings $M_z(R) \cong M_{nz}(R)$, which we may view as $M_z(R) \cong M_n(M_z(R))$. On the surface it might seem that, given proper selection of k_R, $W = M_z(R)$ could itself be a ring of the type described in the Proposition (corresponding to the case $m = 1$). However, such a 'proper selection' of k_R cannot be concocted using the above ideas, as follows. In order to start with $R^{(z)} \cong R^{(nz)}$ (which allows us to get $M_1(W) = W \cong M_n(W)$) we need $nz \equiv z \pmod{k_R}$. On the other hand, to ensure that $W^{(1)}$ is not isomorphic to $W^{(n)}$ we need $n \not\equiv 1 \pmod{k_{M_z(R)}}$; i.e.,

$$n \not\equiv 1 \left(\mathrm{mod}\, \frac{k_R}{\gcd(z, k_R)}\right).$$

But these two congruence statements are incompatible by a standard number theory result. This final remark is meant to give an indication as to why we utilize the somewhat intricate intersection procedure described above.

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Added in proof: Professor G. Bergman has recently demonstrated that the Proposition given above can be significantly strengthened.

PROPOSITION (Bergman): *There exists a ring T for which the matrix rings $M_i(T)$ and $M_j(T)$ are isomorphic for all integers i and j, but for which the free left T-modules $T^{(i)}$ and $T^{(j)}$ are not isomorphic for distinct integers i and j.*

Specifically, let k be a field. For each pair of integers p, q with p dividing q we let r denote the integer q/p, and we let $\tau_{p,q}\colon M_p(k) \to M_q(k)$ denote the map which replaces each entry c of the $p \times p$ matrix A by the $r \times r$ scalar matrix cI_r. (Note: $\tau_{p,q}$ is *not* the scalar embedding of matrix rings utilized above.) Let T denote the direct limit of this system of matrix rings and ring homomorphisms. Then it is not hard to show that T is a ring which possesses the desired properties.

References

[1] G. Abrams, *On the existence of rings R with R isomorphic to $RFM(R)$*, Journal of the Australian Mathematical Society (Series A) **42** (1987), 129–131.

[2] W. Leavitt, *The module type of a ring*, Transactions of the American Mathematical Society **103** (1962), 113–130.